Gray–level Object Segmentation with a Network of FitzHugh-Nagumo Oscillators

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Abstract. In this paper we adopt a temporal coding approach to neuronal modeling of the visual cortex, using oscillations. We propose a hierarchy of three processing modules corresponding to different levels of representation. The first layer encodes the input image (stimulus) into an array of units, while the second layer consists of a network of FitzHugh-Nagumo oscillators. The dynamical behaviour of the coupled oscillators is rigorously investigated and a stimulus-driven synchronization theorem is derived. However, this module reveals itself insufficient to correctly encode and segregate different objects when they have similar gray–levels in the input image. Therefore, a third layer connected in a feedback loop with the oscillators is added. This ensures synchronization (resp. desynchronization) of neuron ensembles representing the same (resp. a different) object. Simulation results are presented using synthetic as well as real and noisy gray–level images.

1 Introduction

A considerable number of neurophysiological findings suggest the hypothesis that brain cells encode information not only by their average firing rate, but also through the precise timing of their firing pattern [7] [14]. This hypothesis has been supported on computational grounds, as one possible way to overcome combinatorial coding strategies, when multiple objects are defined over a multi–dimensional feature space (binding problem) [9] [8]. The basic idea underlying most computational models is that of temporal correlation: two cells encoding the same object establish a relationship by synchronizing their activities, while two cells encoding different objects evolve asynchronously.

In practice, most models employ continuous units exhibiting periodic activity In this case, the problem reduces to finding a network topology capable of imposing appropriate phase shifts in each neuron's activity. The most widely studied unit model is the Wilson–Cowan oscillator, for which a number of analytical results have also been obtained [17] [2]. However, models employing the Wilson–Cowan oscillators appear to have several limitations, in terms of robustness to input noise and in the presence of graded inputs.

In this paper we propose a three–layer architecture based on an alternative neuronal model, the FitzHugh–Nagumo oscillator [6] [12] (cf. figure 1). The

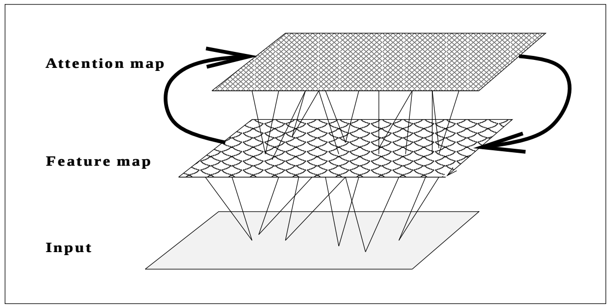

Fig. 1. System architecture.

input layer encodes the stimulus, which is a gray–level image. The second layer is a feature map which consists of an array of FitzHugh–Nagumo oscillators. In section 2 we show that for this model, nice relationships can be established between the value of the external input and the frequency, phase and amplitude of the oscillations. The input from the lower layer is meant to be filtered by some receptive fields, encoding a specific feature, such as edge orientation, color, shape. For the sake of simplicity we shall henceforth consider the trivial receptive field, directly encoding the value of the input unit at the corresponding location. In addition to the external input, the FitzHugh–Nagumo oscillators also receive input from their neighbours (via local coupling). In section 3 we show that, under some constraints on these coupling weights, oscillators receiving the same input asymptotically synchronize, independently of the initial conditions.

This property, although useful to group objects having uniform features, appears insufficient to discriminate two or more objects when they possess similar feature values. This is due to the fact that synchrony depends only on the stimulus intensity but not on the spatial relationships such as connectivity and position in the image. In order to take these spatial relationships into account, a third layer (which we call the attention map) is added, consisting of an array of feature detector units which receive an input signal from the oscillators. The receptive field of these units is implemented as a filter whose impulse response is the difference of two Gaussians having different widths. Its goal is to detect connected groups of oscillators whose average amplitude strongly differs from the background [10]. The attentional signal computed by these units is then fed back to the oscillators as a perturbation, which allows temporary desynchronization of disconnected groups of units.

2 The FitzHugh-Nagumo Neuron Model

Before describing the structure and interconnections of the feature map, we first analyze the model used for the basic units, i.e. the FitzHugh-Nagumo oscillator. This model was directly derived from the Hodgkin–Huxley neuron model and is described in [6] [12]. A neuron's membrane potential is defined by two coupled variables x (excitation variable) and y (recovery variable) whose dynamics is given by:

$$\begin{cases} \epsilon.\frac{dx}{dt} &= -y - g(x) + I \\ \frac{dy}{dt} &= x - by \end{cases} \tag{1}$$

where g is a third order nonlinearity, such as $g(x) = x(x-a)(x-1)$ with $0 < a < 1$, and I is an external input current. The parameter $\epsilon << 1$ controls the speed of variation of the variables x and y, x generally varying faster than y (in which case the system is called a relaxation oscillator). The parameters a and b control the asymptotic behaviour of the system for a given input I as well as the characteristics of the oscillations when they exist (period, amplitude and phase). On the other hand, for fixed a and b, the oscillator dynamics varies according to different values of the input I. Figure 2 (left) shows the system's nullclines for different values of the external input. For more details on the bifurcation analysis of the FitzHugh-Nagumo neuron model, the interested reader is referred to [11] [15].

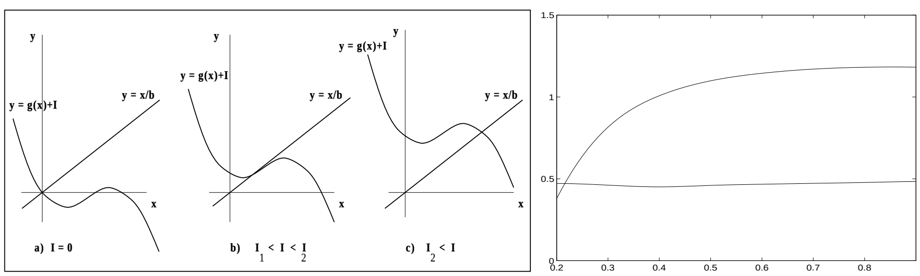

Fig. 2. (left) phase plane nullclines of the FitzHugh-Nagumo system as I varies: In a) $I = 0$ the fixed point is globally stable; in b) the fixed point is unstable and a limit cycle solution is possible; in c) the fixed point is stable again. The system undergoes two Hopf bifurcations as I increases from zero. (right) variation of the amplitude and frequency of oscillation as I varies in a neighborhood of I_1. $a = 0.1,\ b = 0.4$.

Figure 2 (right) shows the variation of amplitude and frequency of oscillations in function of I in the vicinity of the bifurcation. The amplitude is monotonic in I while the frequency is nearly constant. This dependency between the input signal and the amplitude and frequency is particularly important for information encoding. When the input is binary (stimulus present or absent) a correspondence is in general straightforward, since one only needs to have a limit cycle for the stimulus and a fixed point in the absence of a stimulus. To this end, Wilson–Cowan oscillators appear sufficient [2]. However, in our study we are interested in modeling continuous input represented by a gray–level image. In this case, it is essential to establish a one–to–one continuous mapping between the input and the period and/or amplitude of the resulting oscillations. The FitzHugh–Nagumo model satisfies this property, as can be observed from the monotonic dependency between the input and the oscillation amplitude.

3 Synchronization of coupled FitzHugh-Nagumo neurons

In this section, we present the main theoretical result on the stimulus-driven synchronization of coupled FitzHugh-Nagumo neurons. We prove the following theorem in the case of a 1-D chain of oscillators which are symmetrically coupled with nearest–neighbor connections between like x and y units (cf. figure 3).

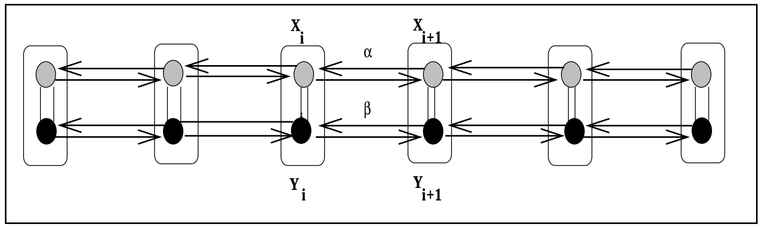

Fig. 3. Connection topology in a chain of FitzHugh-Nagumo oscillators.

Theorem 1. *Given a chain of* N *coupled FitzHugh-Nagumo oscillators* $\{(x_i, y_i)\ ,\ i = 1, ..., N\}$ *receiving the same input* $I_i = I\ ,\ i = 1, ..., N$*, and whose dynamics is described by:*

$$\begin{cases} \frac{dx_i}{dt} = -y_i - g(x_i) + I_i + \alpha(x_{i-1} - x_i) + \alpha(x_{i+1} - x_i)\ ,\ i = 2, ..., N-1 \\ \frac{dy_i}{dt} = x_i - by_i + \beta(y_{i-1} - y_i) + \beta(y_{i+1} - y_i) \end{cases} \tag{2}$$

with boundary conditions:

$$\begin{cases} \frac{dx_1}{dt} = -y_1 - g(x_1) + I_1 + \alpha(x_2 - x_1) \\ \frac{dy_1}{dt} = x_1 - by_1 + \beta(y_2 - y_1) \\ \frac{dx_N}{dt} = -y_N - g(x_N) + I_N + \alpha(x_{N-1} - x_N) \\ \frac{dy_N}{dt} = x_N - by_N + \beta(y_N - y_{N-1}) \end{cases} \tag{3}$$

then the oscillators asymptotically synchronize with zero phase shift provided that α *and* β *satisfy the following conditions:*

$$\begin{cases} \alpha > \frac{1}{2}(1 + 2\beta\cos(\pi/N)) \\ \beta > \frac{1}{2(1-cos(\pi/N))} \\ \beta > \alpha \end{cases} \tag{4}$$

Proof. The basic idea of the proof is to show that $\forall i, r_i^2 = (x_i - x_{i+1})^2 + (y_i - y_{i+1})^2$ tends to zero asymptotically for any initial conditions. Let us first consider the following variable change:

$$x_i - x_{i+1} = r_i\cos(\theta_i)\ ,\ y_i - y_{i+1} = r_i\sin(\theta_i)\ ,\ i = 1, ..., N-1\ ,\ r_i \geq 0\ , \theta_i \in [0, 2\pi[\tag{5}$$

Then, we have:

$$
\begin{aligned}
r_i.\dot{r}_i = \frac{dr_i^2}{2dt} &= \frac{d}{2dt}[(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2] \\
&= (x_i - x_{i+1})(\dot{x}_i - \dot{x}_{i+1}) + (y_i - y_{i+1})(\dot{y}_i - \dot{y}_{i+1}) \\
&= (x_i - x_{i+1})[-(y_i - y_{i+1}) - (g(x_i) - g(x_{i+1})) + \alpha(x_{i-1} - x_i) - 2\alpha(x_i - x_{i+1}) \\
&+ \alpha(x_{i+1} - x_{i+2})] + (y_i - y_{i+1})[(x_i - x_{i+1}) - b(y_i - y_{i+1}) + \beta(y_{i-1} - y_i) \\
&- 2\beta(y_i - y_{i+1}) + \beta(y_{i+1} - y_{i+2})]. \qquad (6)
\end{aligned}
$$

Using simple algebra, it is easy to show that,
$-(x_i - x_{i+1})(g(x_i) - g(x_{i+1})) \leq (x_i - x_{i+1})^2 = r_i^2$.
By assuming $\alpha \leq \beta$, the following inequalities hold:

$$
\begin{aligned}
\alpha\cos(\theta_i)\cos(\theta_{i+1}) + \beta\sin(\theta_i)\sin(\theta_{i+1}) &\leq \beta r_i r_{i+1} \qquad (7) \\
\alpha\cos(\theta_i)\cos(\theta_{i-1}) + \beta\sin(\theta_i)\sin(\theta_{i-1}) &\leq \beta r_i r_{i-1}
\end{aligned}
$$

Therefore, after some developments and simplifications, the previous system can be rewritten in (r_i, θ_i) coordinates as:

$$
\begin{aligned}
r_i.\dot{r}_i &= -r_i\cos(\theta_i)(g(x_i) - g(x_{i+1})) - br_i^2\sin(\theta_i)^2 - 2\alpha r_i^2\cos(\theta_i)^2 - 2\beta r_i^2\sin(\theta_i)^2 \\
&+ r_i r_{i-1}(\alpha\cos(\theta_i)\cos(\theta_{i-1}) + \beta\sin(\theta_i)\sin(\theta_{i-1})) + r_i r_{i+1}(\alpha\cos(\theta_i)\cos(\theta_{i+1}) \\
&+ \beta\sin(\theta_i)\sin(\theta_{i+1})) \\
&\leq r_i^2 - 2\alpha r_i^2 + \beta r_i r_{i-1} + \beta r_i r_{i+1} \qquad (8)
\end{aligned}
$$

Now, let us consider the linear differential system defined by:

$$\dot{q}_i = (1 - 2\alpha)q_i + \beta q_{i-1} + \beta q_{i+1} \quad , \; i = 1, ..., N-1 \qquad (9)$$

which can be rewritten as $\dot{\mathbf{q}} = A\mathbf{q}$ where $\mathbf{q} = (q_1, q_2, ..., q_{N-1})$, and the $N \times N$ matrix A is:

$$
A = \begin{pmatrix}
(1-2\alpha) & \beta & & & \\
\beta & (1-2\alpha) & \beta & & \\
 & \ddots & \ddots & \ddots & \\
 & & \beta & (1-2\alpha) & \beta \\
 & & & \beta & (1-2\alpha)
\end{pmatrix} \qquad (10)
$$

From the inequality (8), we have $\dot{\mathbf{r}} \leq \dot{\mathbf{q}}$ for any initial condition $\mathbf{r}(0) = \mathbf{q}(0)$.Therefore $\mathbf{r}(t)$ is bounded by $\mathbf{q}(t)$. Since $\mathbf{r}(t) \geq 0$, to show that $\mathbf{r}(t)$ tends to zero asymptotically, it suffices to show that this is true for $\mathbf{q}(t)$. The problem hence amounts to find conditions under which the matrix A is stable, i.e. its eigenvalues all have negative real parts. Since A is a triangular Toeplitz matrix [1], its eigenvalues are:

$$\lambda_k = 1 - 2\alpha + 2\beta\cos(k\pi/N) \;, \; k = 1, ..., N-1 \qquad (11)$$

with $\lambda_{N-1} < \lambda_{N-2} < ... < \lambda_1$. To garantee negative eigenvalues, it suffices to impose $1 - 2\alpha + 2\beta\cos(\pi/N) < 0$, i.e. $1 + 2\beta\cos(\pi/N) < 2\alpha$. Since we assumed $\alpha \leq \beta$, this reduces to $1 + 2\beta\cos(\pi/N) \leq 2\beta$, which implies the condition, $\beta \geq \frac{1}{2(1-\cos(\pi)/N)}$, and ends the proof.

The synchronizing dynamics of the system may be intuitively interpreted as follows. For any given initial conditions on (x_i, y_i), the coupling terms $\alpha(x_{i-1} - 2x_i + x_{i+1})$ and $\beta(y_{i-1} - 2y_i + y_{i+1})$, which are proportional to the difference with two neighboring oscillators, are large enough to drive the oscillators towards neighboring fixed points. Eventually, the difference between the oscillators will decrease below some bifurcation threshold. Then, oscillators with identical inputs will adjust minor differences in their phases due to the coupling term to reach zero–lag synchronization.

One can readily see from the constraints of the previous theorem that the values of α and β are monotonically increasing in N. However, simulations revealed that much smaller values are sufficient to achieve synchronization, and that one can safely employ $\alpha < \beta < 1$. Indeed, the above conditions are sufficient but not necessary, and better upper bounds may possibly be found. Similar observations can be made on a result by Campbell and Wang [2] who derived coupling conditions which are sufficient to synchronize a chain of piece-wise linearized Wilson-Cowan oscillators.

4 Simulation Results

To illustrate the theoretical results of the previous section, we present a series of simulations on synthetic and real gray-level images. Figure 4 (left) shows the first image, representing two objects with different gray-levels $I_a = 0.5, I_b = 1.0$ over a background $I_c = 0.0$. Figure 4 (right) shows the network dynamics for

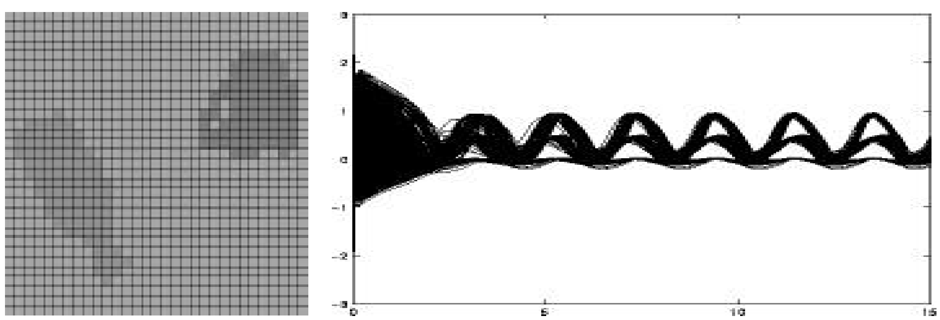

Fig. 4. (left) Synthetic image used for the input layer; (right) temporal diagram of the x variable of the oscillators showing synchronization of two oscillator groups with different amplitudes.

the x_i units from random initial conditions. Each oscillator is connected to its four neighbours in the feature map as in Figure 3. We can observe three groups of oscillators, corresponding to the groups of units receiving the three levels of input. As pointed out earlier, the amplitude of the oscillations characterizes the level of the input. The group with smaller amplitude corresponds to units at the background, as well as units at the frontier of the objects. This simulation illustrates the suitability of the FitzHugh–Nagumo model and the nearest–neighbor connectivity scheme. In particular, this scheme appears capable of grouping together, independently of their initial conditions, units that represent the same, uniform object, provided that its gray level is different from that of other objects.

Another nice property emerges from the next simulation. In this case, two real gray–level images were used, presenting a target object over a very noisy background (cf. figure 4).Thanks to the diffusive local coupling, the array of oscillators appears to increase the contrast between the object and the background, by smoothing regions of high spatial frequency and increasing the amplitude of large groups of neighboring oscillators receiving uniform input.

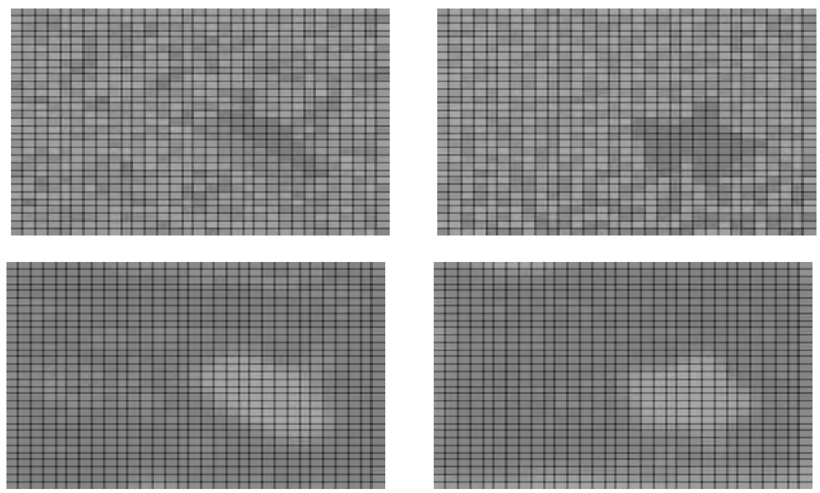

Fig. 5. Segmentation results on real 32×32 noisy images shown at the top. The two images at the bottom represent the corresponding amplitude maps of the 32×32 arrays of oscillators, which show synchronization of oscillators coding the object and desynchronization with the background. Note that due to diffusive local coupling, only large groups of neighbouring oscillators which receive similar input synchronize.

5 Desynchronization by Selective Perturbation

As shown in section 2, the information carried by a FitzHugh–Nagumo oscillator lies in its phase and amplitude, whereas the frequency remains approximatly constant. Also, we have shown that, for the case of nearest–neighbor coupling, the oscillators of *all* units tend to be in phase. This leaves the amplitude as the only information available to separate multiple objects. However, if two disconnected objects have the same intensity in the input image, the two corresponding groups of oscillators will actually synchronize and have the same amplitude. In this situation, the amplitude is no longer sufficient to separate objects.

Therefore, we introduce an *attention* mechanism which carries spatial information on the objects and consequently selectively modifies the phases of the groups of oscillators. Such a mechanism is implemented by the third layer of the architecture (cf. figure 1). It consists of an array of feature detectors which receive at a time T_a (greater than the transient duration) the amplitude map of the array of oscillators.

The feature detectors are defined by the *difference of two Gaussians*, which defines a very commonly found receptive field type in the striate cortex as well as in the thalamus and the superior colliculus. The two latter brain areas have been hypothesized to play a central role in selective attention and eye movements [3] [5]. The goal of the attention layer is to detect blob–like groups of feature map oscillators having similar amplitude to each other, and different amplitude from the surround. Let us consider 2-D arrays of units, of dimension $I \times J$. The activity of unit $a_{i,j}$ at position (i,j) is thus defined by the convolution of the oscillator amplitudes with the difference–of–gaussians filter f:

$$\begin{aligned} a_{i,j} &= \sum_{m=1}^{I} \sum_{n=1}^{J} x_{m,n} \cdot f_{i-m,j-n} \\ f_{i,j} &= \exp - \frac{i^2+j^2}{\sigma^2} - c \cdot \exp - \frac{i^2+j^2}{d\sigma^2}, \end{aligned} \tag{12}$$

where c is a weight guaranteeing equal integrals of the two gaussians over the bounded, discrete image domain, σ is the width of the *on* Gaussian, and d is *off/on* ratio.

Due to the band-pass nature of the filter f, the attention map $\{a_{ij}\}$ will present a number of peaks, in correspondence of regions of oscillators having uniform amplitudes. We can thus represent the object's shape by simply thresholding the values of a, thereby segmenting them from the background. However, we still need to distinguish the selected regions from each other. To this end, it is necessary to encode some spatial position information in an additional layer of units $p_{i,j}, i = 1,\ldots,I; j = 1,\ldots,J$. These units, which we call *perturbation units*, obey the following differential system:

$$\begin{cases} \varepsilon \frac{dp_{i,j}}{dt} = -p_{i,j} + \frac{1}{\sum_{m,n \in N(i,j)} H(a_{m,n})} \sum_{m,n \in N(i,j)} p_{m,n} \cdot H(a_{m,n}) \\ p_{i,j}(0) = \sqrt{i^2+j^2} \cdot H(a_{i,j}) \end{cases}$$

where H is the heaviside threshold function and $N(i,j)$ is the 4–connected neighborhood centered on (i,j). In virtue of the relaxation term combined with the thresholding of the $\{a_{i,j}\}$ values, the above anisotropic diffusion system converges to the following configuration: the value of units located at the background is $p_{i,j} = 0$; units $p_{i,j}$ belonging to the same connected foreground region R will tend to a same constant value, equal to the average value of $\{\sqrt{i^2+j^2}, (i,j) \in R\}$; units located in two different foreground regions will have different values (cf. figure 5 (left)).

After convergence, the different values of the displacement array thus contain the necessary information to distinguish different groups of units. Their values are used as a positive feedback term to the corresponding units $\{(x_{ij}, y_{ij})\}$ in the feature map layer. This feedback acts on the oscillators dynamics as a perturbation which drives the units belonging to the selected regions away from their limit cycles, towards different isochrons. All the units belonging to the same R will thus remain synchronized (because perturbated by the same amount), but will be located in a different location in the phase space from those belonging to another region R'.

Due to the global synchronization properties described above, the effects of the perturbations are limited in time. However, during a transient period, the feature map oscillators will be characterized not only by their amplitude, but also by their phase. We have thus obtained a way to discriminate, through temporary perturbations, different objects even if they receive equal external inputs.

Figure 5 (right) shows the temporal behavior of three groups of oscillators, with the perturbation occurring at $t = 9$. In this case, the input signal is the same of figure 4, although the two foreground regions have now been set to the same value. It can be noticed that, without altering the dynamics of the background oscillators, the units of the two other groups are shifted to different phases.

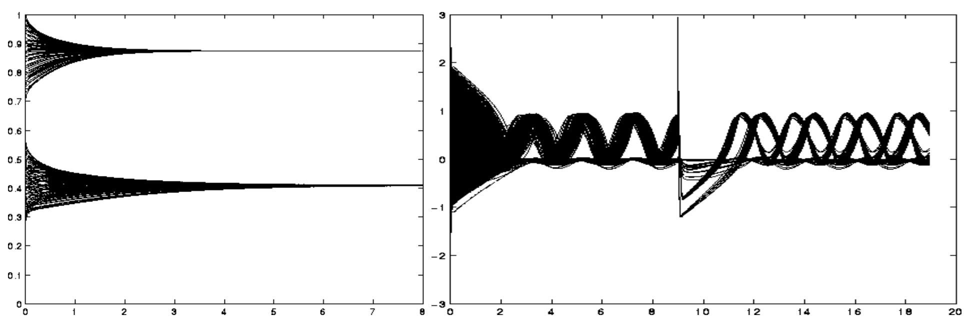

Fig. 6. Temporal behaviour of units of the attentional layer (left) and oscillators of the feature map (right) before and after perturbation at $t = 9$.

6 Discussion

The present paper adresses a central issue in the temporal correlation approach to visual modeling, which concerns gray-level image segmentation using continuous oscillators. Several previous studies have adressed the same issue, among them [13] and [16] used networks of Wilson-Cowan oscillators. However, their architecture is quite complex since the input stimulus is encoded by employing as many networks of oscillators as there are gray levels in the input. In fact these oscillators receive only binary inputs whereas FitzHugh-Nagumo units described in this paper establish a one to one mapping between a graded input and the amplitude and frequency of the oscillations. We have shown nice synchronization properties of FitzHugh-Nagumo oscillators, using simple nearest-neighbor connections. We also introduced a selective perturbation approach to desynchronization of oscillator groups encoding different objects. The basic ide is to encode a spatial feature in the oscillators dynamics as a perturbation which drives disconnected groups of oscillators to different isochrons in the phase space. Several lines of improvements are under investigation. The isotropic connectivity between oscillators should be ameliorated in a way which reflects the contrast variation in the input image. This would require a dynamic coupling approach based on anisotropic reaction-diffusion operators. We are also analysing the isochrons of the system (2) in order to establish a relationship between the perturbation values and the phases of oscillators.

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